Unit 2: Numerical Methods

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Introduction

Numerical methods are techniques used to obtain approximate solutions to mathematical problems that are otherwise difficult or impossible to solve analytically. They are essential in engineering, physics, and applied sciences, especially when equations or data cannot be handled symbolically.

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1 Roots of Equations

The **root of an equation** $f(x) = 0$ is the value of x for which $f(x)$ becomes zero. Numerical methods are used when algebraic solutions are not possible.

1.1 Quadratic Formula

For $ax^2 + bx + c = 0$, the roots are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example: Solve $2x^2 + 5x - 3 = 0$.
Solution:

$$x = \frac{-5 \pm \sqrt{25 - 4(2)(-3)}}{4} = \frac{-5 \pm \sqrt{49}}{4} = \frac{-5 \pm 7}{4}.$$

Hence, $x = \frac{1}{2}$ or $x = -3$.

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1.2 Iterative Method

If the equation can be written as $x = g(x)$, then successive approximations can be obtained by:

$$x_{n+1} = g(x_n).$$

Example: Solve $x = \cos x$ using iteration starting with $x_0 = 0.5$.

Solution:

$$x_1 = \cos(0.5) = 0.8776, \quad x_2 = \cos(0.8776) = 0.6390, \quad x_3 = \cos(0.6390) = 0.8027.$$

Continue until $|x_{n+1} - x_n|$ is small. The root is approximately $x \approx 0.739$.

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1.3 Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Example: Find the root of $f(x) = x^2 - 5$ starting with $x_0 = 2$.

Solution:

$$f(x) = x^2 - 5, \quad f'(x) = 2x.$$

$$x_1 = 2 - \frac{2^2 - 5}{4} = 2.25, \quad x_2 = 2.25 - \frac{2.25^2 - 5}{4.5} = 2.2361.$$

Hence, the root is approximately $\sqrt{5} = 2.236$.

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1.4 Bisection Method

Steps:

1. Choose a, b such that $f(a)$ and $f(b)$ have opposite signs.
2. Compute midpoint $c = (a + b)/2$.
3. Evaluate $f(c)$ and replace a or b accordingly.

Example: Find a root of $f(x) = x^3 - 4x - 9 = 0$ between 2 and 3.

Solution: $f(2) = -9$, $f(3) = 6$, so the root lies between 2 and 3. Midpoint $c = 2.5$, $f(2.5) = -1.375$. New interval: $(2.5, 3)$. Repeat to get the root ≈ 2.7.

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1.5 Regula-Falsi (False Position) Method

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

Example: Solve $f(x) = x^3 - 5x + 1 = 0$ between 0 and 1.

Solution: $f(0) = 1$, $f(1) = -3$.

$$x = \frac{0(-3) - 1(1)}{-3 - 1} = 0.25.$$

Evaluate $f(0.25) = 0.25^3 - 5(0.25) + 1 = -0.9375$. The root lies between 0 and 0.25. Continue iteration to approximate $x \approx 0.2$.

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2 Differential Calculus: Numerical Differentiation

$$f'(x_0) \approx \frac{f(x_1) - f(x_0)}{h}, \quad f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h}.$$

Example: Given $f(1) = 2.7183, f(1.1) = 3.0042, f(1.2) = 3.3201$, find $f'(1.1)$ using central difference.

Solution:

$$f'(1.1) \approx \frac{f(1.2) - f(1.0)}{2(0.1)} = \frac{3.3201 - 2.7183}{0.2} = 3.009.$$

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3 Integral Calculus: Numerical Integration

3.1 Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2(f(x_1) + f(x_2) + \dots) + f(b)].$$

Example: Compute $\int_0^1 x^2 dx$ using $n = 4$ intervals.

Solution: $h = 0.25$, and $f(x) = x^2$.

$$I = \frac{0.25}{2}[f(0) + 2(f(0.25) + f(0.5) + f(0.75)) + f(1)] = 0.3333.$$

Exact value $= 1/3 = 0.3333$.

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3.2 Simpson's 1/3 Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + \dots + f(x_n)].$$

Example: Find $\int_0^1 \frac{1}{1+x^2} dx$ using $n = 4$.

Solution: $h = 0.25$, compute $f(0) = 1, f(0.25) = 0.9412, f(0.5) = 0.8, f(0.75) = 0.64, f(1) = 0.5$.

$$I = \frac{0.25}{3}[1 + 4(0.9412) + 2(0.8) + 4(0.64) + 0.5] = 0.7854.$$

Exact value $= \tan^{-1}(1) = \pi/4 = 0.7854$.

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3.3 Probability Distributions and Mean Value

If $f(x)$ is the probability density function,

$$E[X] = \int_a^b x f(x)\, dx.$$

Example: If $f(x) = 3x^2$, $0 \le x \le 1$, then

$$E[X] = \int_0^1 3x^3 dx = \frac{3}{4} = 0.75.$$

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4 Simultaneous Equations and Matrices

4.1 Matrix Operations

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.$$

$$A + B = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}, \quad AB = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}.$$

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4.2 Gauss-Seidel Method

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{(k+1)} - \sum_{j>i} a_{ij} x_j^{(k)} \right).$$

Example: Solve

$$10x + y + z = 12, \quad 2x + 10y + z = 13, \quad 2x + 2y + 10z = 14.$$

Start with $x = y = z = 0$.

Iteration 1:

$$x^{(1)} = (12 - 0 - 0)/10 = 1.2,$$

$$y^{(1)} = (13 - 2(1.2) - 0)/10 = 1.06,$$

$$z^{(1)} = (14 - 2(1.2) - 2(1.06))/10 = 1.048.$$

Repeat until convergence: $x = 1, y = 1, z = 1$ (approx).

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5 Interpolation, Extrapolation, and Curve Fitting

5.1 Interpolation (Lagrange's Formula)

$$f(x) = f(x_0)\frac{(x - x_1)}{(x_0 - x_1)} + f(x_1)\frac{(x - x_0)}{(x_1 - x_0)}.$$

Example: Given $f(1) = 1$, $f(2) = 4$, estimate $f(1.5)$.

$$f(1.5) = 1\frac{(1.5 - 2)}{(1 - 2)} + 4\frac{(1.5 - 1)}{(2 - 1)} = 2.5.$$

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5.2 Curve Fitting (Least Squares Method)

We fit a line $y = a + bx$ such that

$$a = \frac{\sum y - b\sum x}{n}, \quad b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}.$$

Example: For $(x, y) : (1, 2), (2, 3), (3, 5)$,

$$\sum x = 6,\ \sum y = 10,\ \sum xy = 23,\ \sum x^2 = 14,\ n = 3.$$

$$b = \frac{3(23) - 6(10)}{3(14) - 36} = 1.5, \quad a = \frac{10 - 1.5(6)}{3} = 0.$$

Hence, best-fit line: $y = 1.5x$.

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Exercises

1. Use the Newton-Raphson method to find $\sqrt{3}$ correct up to three decimal places.

2. Solve $x^3 - x - 2 = 0$ by the Bisection method.

3. Use the Trapezoidal rule with $n = 4$ to evaluate $\int_0^2 (x^2 + 1)dx$.

4. Using central differences, find $f'(x)$ at $x = 1$ for $f(x) = e^x$ from $f(0.9) = 2.46$, $f(1.0) = 2.72$, $f(1.1) = 3.00$.

5. Solve the system:

$$4x + y + z = 7, \quad x + 5y + z = 8, \quad 2x + 3y + 10z = 12$$

   by the Gauss-Seidel method.

6. Fit a straight line $y = a + bx$ to the data: $(1, 1.2)$, $(2, 1.9)$, $(3, 3.0)$, $(4, 3.8)$.

7. Using Simpson's rule, evaluate $\int_0^2 \frac{1}{1+x^2} dx$ with $n = 4$.

Conclusion

Numerical methods form the basis for computer-based problem solving in applied mathematics. They allow approximate but highly accurate solutions to problems where analytical methods are impractical.